Kinetic equation for classical particles obeying an exclusion principle

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In this paper we analyze the kinetics of classical particles which obey an exclusion principle (EP) in the only-individual-transitions (OIT) approximation, and separately in the more rigorous contemporary-transitions (CT) description. In order to be able to include the EP into the kinetics equations we consider a discrete, one-dimensional, heterogeneous and anisotropic phase space and, after defining the reduced transition probabilities, we write a master equation. As a limit to the continuum of this master equation we obtain a generalized Fokker-Planck (FP) equation. This last is a nonlinear partial differential equation and reduces to the standard FP equation if the nonlinear term, which takes into account the EP, is neglected. The steady states of this equation, both in the OIT approximation and CT description, are considered. In the particularly interesting case of Brownian particles as a steady state in the OIT approximation we obtain the Fermi-Dirac (FD) distribution, while in the CT description we obtain another distribution which differs slightly from that of the FD. Moreover, our approach permits us to treat in an alternative and efficient way the problem of the determination of an effective potential to simulate the exclusion principle in classical many-body equations of motion.

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I. INTRODUCTION

The Fokker-Planck (FP) equation is widely used in many different fields of research in classical and quantum physics such as, for example, the diffusion of particles (molecules, atoms, ions, etc.) in condensed matter [1–3], the transport of charged particles in semiconductors [4,5], the kinetics of particles in a plasma [6–8], and the nuclear fission and the phase transitions in high-energy physics [9–12]. Many authors have recently analyzed different mathematical procedures to solve exactly this equation in the framework of particular models [13–17]. The FP equation can be seen as the limit into the continuum of the master equation of a Markovian process [18–21].

In a previous work [17] we have solved the FP equation with the coefficients of friction and diffusion expanded in polynomial series with positive coefficients. Stationary solutions for systems of classical particles are Maxwellian and non-Maxwellian distribution functions. The results we have obtained are in agreement with those derived by Zhu [22]. In his work the dynamics of the particle motion is governed by a nonlinear Langevin equation while the probability density of the stochastic variables is obtained from a nonlinear FP equation.

In this paper we pay attention to the problem of evolution in time of the particles of a physical system, obeying an exclusion principle (EP), the motion being described by a FP equation or by a master equation in which an EP has been included. An example is the hopping transport on a lattice of ionic conductors: two ions having the same charge cannot occupy the same site due to their natural electrostatic repulsion. A second example is given by the diffusion of molecules of type A in a lattice of molecules of type B which react with one single molecule A at a time: in this case the kinetics of molecules A obeys an EP.

Many authors have studied the kinetics on a lattice of particles obeying a particular exclusion principle by means of the only-individual-transitions (OIT) approximation [23–26] (in a time interval dt only one transition is allowed). This approximation is correct if the gas of particles is rarefied and it is incorrect if the gas is dense. Wilets and co-workers [27–29] and Dorso, Duarte, and Randrup [30] have discussed the incorporation approximately of the EP into equations of motion for classical particles in order to develop a model for heavy-ion collisions and a classical simulation of the Fermi gas. We wish to mention that Vassiliev [31] has taken into account approximately the EP in the collision term of the Boltzmann equation.

We want to analyze, in the space of velocities, how we can introduce an EP in the particle kinetics and to look for the stationary solutions of the kinetics differential equation. For this purpose we must consider the discrete space of velocities. Then we may introduce an EP in a correct way.

We have taken into account that two particles occupying, at the same time, two different sites $i\pm1$ could make a transition to the site i; but of course, due to the EP, one of the two possible transitions is forbidden. In what follows we call this scheme the contemporary transitions (CT) description.

Then we return to the continuum obtaining a nonlinear, partial differential equation. The stationary solution is a distribution function which differs very slightly from the FD distribution in the case of the CT description, while in the OIT description we obtain exactly the FD distribution. This result is not too surprising. In fact, we know that the FP equation, which is a linear differential equation for the probability density, describes a Markovian stochastic process. One can show that the quantum-mechanics formulation based on the Schrödinger equation is conceptually comparable to the description of a classical Markovian stochastic process based on the FP equation, which is formally very similar to the Schrödinger equation. One can interpret the quantum physics as a particular stochastic classical process and reformulate the quantum mechanics in terms of the classical probabilistic concepts, as discussed in the past by Nelson [32].

The introduction of an EP into the FP equation allows us to obtain either the FD distribution or an alternative distribution very close to the FD one in place of a classical (Maxwellian or non-Maxwellian) distribution when the EP is not introduced (plasma conditions for non-Maxwellian electron distributions have been studied recently [33]).

We illustrate the main purposes and results of this paper by describing the contents of the different sections.

In Sec. II we consider a discrete one-dimensional heterogeneous and anisotropic phase space which can be identified with a Markovian chain. The evolution in time of the system is described by the mean-field approximation [26] with the transition probabilities written so as to include an EP.

In Sec. III we consider a continuous one-dimensional heterogeneous and anisotropic phase space. We derive, for particles rigorously obeying the EP, a partial differential equation which can be viewed as a generalized FP equation. We also deduce the kinetics equation in the particularly interesting case that the space is homogeneous and isotropic.

In Sec. IV we deduce the stationary states both from the equation of the OIT approximation and from the equation derived in Sec. III where the EP is more properly considered.

In Sec. V we derive the equations describing the kinetics of Brownian particles obeying an EP.

In Sec. VI we obtain from the kinetics equations of Brownian particles in the OIT approximation the stationary distribution which turns out to have the expression of a FD distribution. Furthermore, using our rigorous approach described in Sec. III, we obtain a statistical distribution which differs slightly from the FD one, but when the value of a certain parameter goes to zero, or when the temperature goes asymptotically to zero, this statistical distribution equals that of the FD.

In Sec. VII we describe the particle kinetics from the point of view of the repulsive potential introduced by the EP and relate the present approach to previous works by Wilets and co-workers [27–29] and Dorso, Duarte, and Randrup [30].

Finally, conclusions are outlined in Sec. VIII.

II. THE MASTER EQUATION

Let us consider the kinetics of N particles in a one-dimensional discrete phase space (identified by the velocity space), which is a one-dimensional Markovian chain. The generic site is labeled by the index i ($i=0,\pm1,\pm2,\ldots$); the velocity at the ith site is $v_i=i\Delta v$, where Δv is a constant that exists without dimensions.

We call $p_i(t)$ the occupational probability of the ith site, satisfying the condition $0\le p_i(t)\le 1$. We assume that a particle at site i at time t can hop to site j at time $t+dt$ with a transition probability $W_{ij}(t)$, and in the time interval dt at most N particles can make a transition to the nearest site. Therefore we define

$$W_{ij}(t)=\frac{c}{\Delta v^2}[w_i^+(t)\delta_{ji+1}+w_i^-(t)\delta_{ji-1}]\ , \tag{1}$$

where c is a dimensional parameter that we assume to be equal to one. We define the reduced transition probability $\pi_i^+(t)$:

$$\pi_i^{\pm}(t)=\frac{w_i^{\pm}(t)}{\Delta v^2}p_i(t) \tag{2}$$

depending on the particle interactions.

The Markovian master equation can be written as follows:

$$\frac{dp_i(t)}{dt}=\pi_{i-1}^+(t)+\pi_{i+1}^-(t)-\pi_i^+(t)-\pi_i^-(t)\ . \tag{3}$$

The meaning of this equation is clearly exemplified in Fig. 1. We must now explicitly state the reduced transition probability.

The main feature of the OIT approximation is given by the condition that only one particle at a time can be free to (though it does not necessarily) hop. By adding to this condition the requirement that the particles obey an EP, we can write

$$\pi_i^{\pm}(t)=\frac{\alpha_i^{\pm}(t)}{\Delta v^2}p_i(t)[1-p_{i\pm1}(t)]\ . \tag{4}$$

The factor $[1-p_{i\pm1}(t)]$ means that only the transitions to the empty $i\pm1$ site are allowed and they are forbidden if the site is occupied. The hypothesis of individual transition is correct if the gas of particles distributed along the Markovian chain is very dilute. A more complete expression of the function $\pi_i^{\pm}(t)$ can be obtained by taking into account contemporary transitions, which also must obey an EP.

We assume that the transition to the empty site $i\pm1$ is

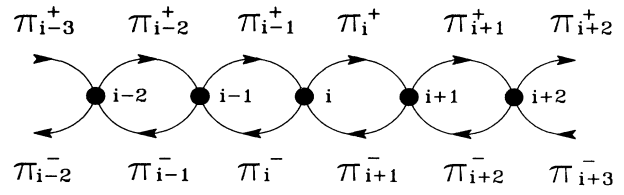

FIG. 1. One-dimensional Markovian chain where the transition probabilities are indicated.

allowed if, at the same time, the transition $i\pm2\rightarrow i\pm1$ is forbidden. Then we may write the following expression for the reduced transition probability:

$$\pi_i^{\pm}(t)=\frac{\alpha_i^{\pm}(t)}{\Delta v^2}p_i(t)[1-p_{i\pm1}(t)] \times\{1-\alpha_{i\pm2}^{\mp}(t)p_{i\pm2}(t)[1-p_{i\pm1}(t)]\} . \quad (5)$$

It is clear from Eq. (5) that in addition to the transitions to the nearest sites $i\rightarrow i\pm1$ we consider, at the same time, the transitions of the particles belonging to the next site $i\pm2$. In this way we rigorously take the EP into account. Equation (5) is the main important relation of the approach to classical particle kinetics, described in this work. We call it the CT description. Therefore the ensemble of the above equations is a system of equations describing the kinetics of N interacting particles obeying the EP.

If we define

$$j_i(t)=[\pi_i^{+}(t)-\pi_{i+1}^{-}(t)]\Delta v , \quad (6)$$

which represents a discrete current, the master equation can be written as

$$\frac{dp_i(t)}{dt}+\frac{\Delta j_i(t)}{\Delta v}=0 , \quad (7)$$

where

$$\Delta j_i(t)=j_{i+1}(t)-j_i(t) . \quad (8)$$

Equation (7) represents a continuity equation.

III. KINETICS EQUATIONS

We derive in the continuum limit a partial differential equation for the kinetics of particles rigorously obeying the EP (the CT description). The one-dimensional continuous phase space is the limit of the discrete space of the Markovian chain when $\Delta v\rightarrow0$. Let us define the new quantities of interest for the continuous phase space:

$$p(t,v)=\lim_{\Delta v\rightarrow0}p_i(t) , \quad (9)$$

$$\alpha^{\pm}(t,v)=\lim_{\Delta v\rightarrow0}\alpha_i^{\pm}(t) , \quad (10)$$

$$j(t,v)=\lim_{\Delta v\rightarrow0}j_i(t) . \quad (11)$$

The physical process of the particles transport in the continuous phase space is a Markovian process. The hypothesis that in the time interval dt only individual transitions from site i to site $i\pm1$ occur, while valid in the discrete space, can be expressed in the continuous as follows: the probability $p(t,v)$ becomes $p(t+dt,v\pm dv)$ at $t+dt$. This means that we do not take into account the sudden variation of velocities. However, in the case of the Boltzmann equation, the particle collisions force us to consider these variations. The master equation (7) becomes, in the limit $\Delta v\rightarrow0$,

$$\frac{\partial p(t,v)}{\partial t}+\frac{\partial j(t,v)}{\partial v}=0 , \quad (12)$$

which is the continuity equation.

The expression of the current $j(t,v)$ of particles obeying the EP can be obtained using Eqs. (5), (6), and (9)–(11):

$$j(t,v)=\frac{1}{dv}(\alpha^{+}(t,v)P(t,v)[1-p(t,v+dv)]\{1-\alpha^{-}(t,v+2dv)p(t,v+2dv)[1-p(t,v+dv)]\} -\alpha^{-}(t,v+dv)p(t,v+dv)[1-p(t,v)]\{1-\alpha^{+}(t,v-dv)p(t,v-dv)[1-p(t,v)]\}) . \quad (13)$$

We expand to the first order in dv the quantities that depend on dv and Eq. (13) becomes

$$j(t,v)=\frac{1}{dv}[\alpha^{+}(t,v)-\alpha^{-}(t,v)]p(t,v)[1-p(t,v)]-\left[\frac{\partial\alpha^{-}(t,v)}{\partial v}p(t,v)[1-p(t,v)]+[\alpha^{+}(t,v)-\alpha^{-}(t,v)]p(t,v)\frac{\partial p(t,v)}{\partial v} +\alpha^{-}(t,v)\frac{\partial p(t,v)}{\partial v}+\frac{\partial}{\partial v}[\alpha^{-}(t,v)\alpha^{+}(t,v)]p^2(t,v)[1-p(t,v)]^2 -4\alpha^{-}(t,v)\alpha^{+}(t,v)p(t,v)[1-p(t,v)][p(t,v)-\tfrac{1}{2}]\frac{\partial p(t,v)}{\partial v}\right] . \quad (14)$$

After introducing the functions $D(t,v)$ and $J(t,v)$ defined as usual by

$$\alpha^{+}(t,v)-\alpha^{-}(t,v)=J(t,v)dv , \quad (15)$$

$$\alpha^{+}(t,v)+\alpha^{-}(t,v)=D(t,v) , \quad (16)$$

we can write the expression $j(t,v)$ for the current

$$j(t,v)=-\left[\tfrac{1}{2}D(t,v)\frac{\partial p(t,v)}{\partial v}+\left[-J(t,v)+\frac{1}{2}\frac{\partial D(t,v)}{\partial v}\right]p(t,v)[1-p(t,v)]+\frac{\partial}{\partial v}\{\tfrac{1}{2}D(t,v)p(t,v)[1-p(t,v)]\}^2\right] . \quad (17)$$

Finally the continuity equation (12) becomes

$$\frac{\partial p(t,v)}{\partial t}=\frac{\partial}{\partial v}\left[\tfrac{1}{2}D(t,v)\frac{\partial p(t,v)}{\partial v}+\left[-J(t,v)+\frac{1}{2}\frac{\partial D(t,v)}{\partial v}\right]p(t,v)(1-p(t,v)+\frac{\partial}{\partial v}\{\tfrac{1}{2}D(t,v)p(t,v)[1-p(t,v)]\}^2\right] . \quad (18)$$

Equation (18) is a nonlinear, partial differential equation of first order in the time variable and of second order in the velocity variable, and describes a Markovian process. The function $p(t,v)$ can be given at any t by solving Eq. (18) if we know its value $p_0(v)$ at the initial time. We define $p(0,v)=p_0(v)$.

We will now deduce the kinetics equation in a particular case. We assume an homogeneous and isotropic space and thus we have

$$D(t,v)=D , \quad (19)$$

$$J(t,v)=0 . \quad (20)$$

With the substitution $Dt\rightarrow t$, Eq. (18) becomes

$$\frac{\partial p(t,v)}{\partial t}=\frac{\partial^2}{\partial v^2}\{p(t,v)+\tfrac{1}{2}Dp^2(t,v)[1-p(t,v)]^2\} . \quad (21)$$

If we define the diffusion coefficient as a function of p in this way:

$$D(p)=1+Dp(1-p)(1-2p) , \quad (22)$$

then Eq. (21) assumes the expression:

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial v}\left[D(p)\frac{\partial p}{\partial v}\right] \quad (23)$$

and describes the kinetics of particles obeying the EP as a nonlinear diffusion, with a diffusion coefficient depending on the function $p(t,v)$. Classical particles not obeying an EP have the Maxwell-Boltzmann distribution function. Classical particles obeying an EP, such as the Pauli EP, have a distribution function which coincides with that of the FD (see Sec. VI).

The following case shows the equations and the results that we obtain in the OIT approximation. The current is

$$j(t,v)=\frac{1}{dv}\{\alpha^+(t,v)p(t,v)[1-p(t,v+dv)] -\alpha^-(t,v+dv)p(t,v+dv)[1-p(t,v)]\} \quad (24)$$

and with the same approach used to derive Eq. (18) we finally obtain the equation

$$\frac{\partial p(t,v)}{\partial t}=\frac{\partial}{\partial v}\left[\frac{1}{2}D(t,v)\frac{\partial p(t,v)}{\partial v}+\left[-J(t,v)+\frac{1}{2}\frac{\partial D(t,v)}{dv}\right]\times p(t,v)[1-p(t,v)]\right] . \quad (25)$$

If the space is homogeneous and has properties independent of time we have

$$D(t,v)=D , \quad (26)$$

$$J(t,v)=J , \quad (27)$$

and Eq. (25) becomes

$$\frac{\partial p(t,v)}{\partial t}=\frac{\partial}{\partial v}\left[\frac{1}{2}D\frac{\partial p(t,v)}{\partial v}-Jp(t,v)[1-p(t,v)]\right] , \quad (28)$$

which coincides with the well-known Burgers equation [34,35]. Equations (18) and (25) change to the following FP equation for classical particles if the nonlinear terms, which take into consideration the EP, are neglected:

$$\frac{\partial p(t,v)}{\partial t}=\frac{\partial}{\partial v}\left[-J(t,v)p(t,v)+\frac{1}{2}\frac{\partial}{\partial v}[D(t,v)p(t,v)]\right] . \quad (29)$$

This equation can be directly derived using the procedure to get Eq. (18) if we pose in Eq. (2), $w_i^{\pm}(t)=\alpha_i^{\pm}(t)$.

IV. STATIONARY STATES

We show, in this section, how stationary solutions can be derived from the kinetics equations. We remember that the stationary-state distribution satisfying Eq. (12) can be deduced by solving the equation $j(\infty,v)=0$.

The stationary distribution for particles obeying the EP in the CT description can be obtained from Eq. (17) by setting $j(\infty,v)=0$. We obtain the nonlinear differential equation

$$\frac{\partial p}{\partial v}=\frac{\left[2J-\frac{\partial D}{\partial v}\right]p(1-p)-D\frac{\partial D}{\partial v}p^2(1-p)^2}{D+D^2p(1-p)(1-2p)} . \quad (30)$$

Equation (30) can be easily solved when D is a constant. Easy manipulations give

$$D\ln\frac{1-p}{p}+D^2p(1-p)=-2\int J(v)dv+\mu' , \quad (31)$$

where μ' is the integration constant to be determined from the condition

$$\int_0^{+\infty}p(E)dE=E_F , \quad (32)$$

where $E=\frac{1}{2}mv^2$ is the particle kinetic energy and E_F is the Fermi energy level. The distribution function p we are looking for is therefore the solution of Eq. (31). In the case of the OIT approximation, by posing $j(\infty,v)=0$ in Eq. (25), we have

$$\frac{\partial p}{\partial v}=\left[\frac{2J}{D}-\frac{1}{D}\frac{\partial D}{\partial v}\right]p(1-p) . \quad (33)$$

Equation (33) can be easily integrated:

$$p(v)=\left[1+D(v)\exp\left\{-2\int\frac{J(v)}{D(v)}dv+\bar{\mu}\right\}\right]^{-1} \quad (34)$$

and the integration constant $\bar{\mu}$ can be determined from condition (32). Finally when the kinetics is described by the classical FP equation (29) we have

$$p(v)=p_0\frac{1}{D(v)}\exp\left\{2\int\frac{J(v)}{D(v)}dv\right\} . \tag{35}$$

V. BROWNIAN MOTION

In the case of Brownian motion we derive three different equations. The first describes the motion without the inclusion of an EP, the second describes the motion in the OIT approximation with an EP, and the third is related to the motion in the CT description. The three distribution functions for the three cases will be given in Sec. VI.

First let us consider a Brownian classical particle moving without the action of an EP [36–38] and subjected to a random force with a white spectrum, and coming from a parabolic potential. It is well known that in this case the function $p(t,v)$ is a solution of a FP equation with the diffusion coefficient $D(t,v)$ and the drift coefficient $J(t,v)$ given by the expressions

$$J(t,v)=-\gamma v , \tag{36}$$

$$D(t,v)=\frac{2kT}{m}\gamma , \tag{37}$$

where γ is a constant, k is the Boltzmann constant, T is the temperature, and m is the mass of the particles. After the substitutions

$$\gamma t\rightarrow t , \tag{38}$$

$$\sqrt{m/kT}\,v\rightarrow v , \tag{39}$$

$$\gamma kT/m\rightarrow\alpha , \tag{40}$$

the FP equation becomes

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial v}\left[\frac{\partial p}{\partial v}+vp\right] . \tag{41}$$

In the second case the kinetics of Brownian particles obeying the EP in the OIT approximation is described by Eq. (25), which can be written

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial v}\left[\frac{\partial p}{\partial v}+vp(1-p)\right] \tag{42}$$

after having taken into account Eqs. (36)–(39). Finally, the differential equation for Brownian motion of particles obeying a more rigorous version of the EP (the CT description) is, considering Eqs. (18) and (36)–(40),

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial v}\left[\frac{\partial p}{\partial v}+vp(1-p)+\alpha\frac{\partial}{\partial v}[p(1-p)]^2\right] . \tag{43}$$

VI. STATISTICAL DISTRIBUTIONS

We can deduce from Eq. (41) (where the EP is not included) the distribution function of Brownian particles; it is the well-known Maxwell-Boltzmann distribution:

$$p(E)=p_0\exp\left\{-\frac{E}{kT}\right\} . \tag{44}$$

In the OIT approximation, the distribution function of Brownian classical particles obeying the EP can be deduced from Eq. (43) or (34) by taking into account Eqs. (36) and (37). We obtain

$$p(E)=\left[1+\exp\left\{\frac{E-\mu}{kT}\right\}\right]^{-1} , \tag{45}$$

which is, in fact, the FD distribution.

From Eq. (43), which takes into account the EP more rigorously (the CT description) than in the OIT approximation, we obtain the statistical distribution implicitly defined by the relation

$$\frac{E-\mu'}{kT}=\ln\frac{1-p(E)}{p(E)}+\alpha p(E)[1-p(E)] . \tag{46}$$

The second term on the right-hand side of Eq. (46) is a correction of the FD distribution. The above distribution differs slightly from the FD one, though it equals it if the parameter α goes to zero.

We remark that the distribution function of classical particles which do not obey an EP is the Maxwell-Boltzmann distribution which differs, of course, from the Bose-Einstein (BE) distribution of quantum bosons. Introducing an inclusion principle in the kinetics equations, one can derive the BE distribution (this argument will be treated elsewhere). On the other hand, classical particles obeying an EP, such as the Pauli exclusion principle, are described by a distribution function which coincides with the FD distribution of quantum fermions.

We now calculate the constants μ and μ' using Eq. (32). These quantities assume the role of chemical potentials and are functions of the temperature. We define the quantity ξ.

$$\xi=\xi(T,E_F,\alpha)=p(0) , \tag{47}$$

which always has a value less than one; we also set

$$\xi_0=\xi(T,E_F,0) . \tag{48}$$

In the case of the FD distribution (Eq. 45) we have

$$\xi_0=1-\exp(-E_F/kT) \tag{49}$$

and

$$\mu(T)=kT\ln[\exp(-E_F/kT)-1] . \tag{50}$$

In the case of the distribution given by Eq. (46) we have $\xi(T,E_F,\alpha)$ implicitly defined by the equation

$$\ln\left[\frac{1-\xi}{1-\xi_0}\right]=\tfrac{1}{6}\alpha\xi^2(4\xi-3) \tag{51}$$

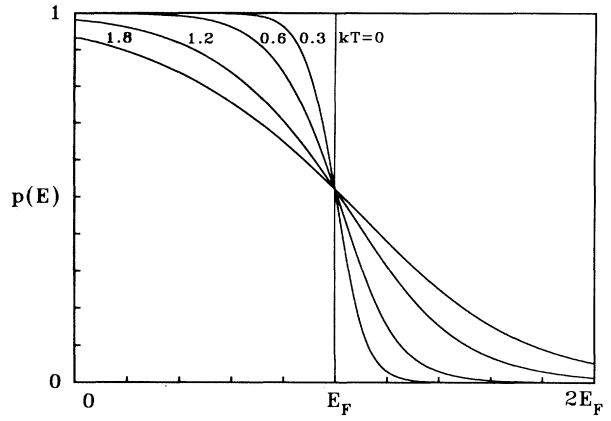

FIG. 2. Distribution function in the CT description when $\alpha=1$, at different temperatures.

and

$$\mu'(T)=\mu(T)-\Delta\mu(T) , \tag{52}$$

where

$$\Delta\mu(T)=kT\left[\alpha\xi(1-\xi)+\ln\frac{\xi_0(1-\xi)}{\xi(1-\xi_0)}\right] . \tag{53}$$

We remark that if $\alpha\rightarrow 0$, then $\Delta\mu(T)\rightarrow 0$ and $\mu'(T)=\mu(T)$: furthermore, the distribution given by Eq. (46) becomes that of the FD. If the temperature T goes asymptotically to zero, both the functions given by Eqs. (45) and (46) become the same standard step distribution, so we have $\mu(0)=\mu'(0)=E_F$.

In Fig. 2 we report the distribution given by Eq. (46) at the value $\alpha=1$ and for several temperatures. In Fig. 3 we compare the distribution derived in the CT description given by Eq. (46) to that of the FD at two different temperatures. We remark that the difference between the two distributions vanishes when $T\rightarrow 0$.

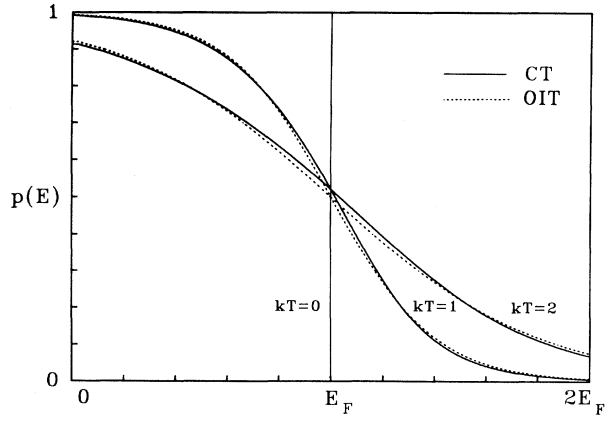

FIG. 3. Comparison, at two different temperatures, of the distribution function in the CT description (continuous lines) with the one in the OIT approximation (dashed lines).

VII. THE POTENTIAL INTRODUCED BY THE EP

We now wish to discuss, in the framework of the OIT approximation, the effect of the EP with particular attention to the potential introduced by the EP itself. We relate the present approach to previous work by Wilets and co-workers [27–29] and Dorso, Duarte, and Randrup [30] where these authors introduced a particular potential into the classical equations to simulate the Pauli EP. We can also show that our approach is equivalent to the introduction of a repulsive Pauli potential.

It is well known that if we introduce the potential $U(v)$ by means of the relation

$$\frac{\partial U}{\partial v}=\frac{1}{D}\left[\frac{\partial D}{\partial v}-2J\right] , \tag{54}$$

the FP equation for particles not obeying the EP can be written in the form

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial v}\left[\frac{D}{2}\left[\frac{\partial U}{\partial v}p+\frac{\partial p}{\partial v}\right]\right] . \tag{55}$$

In the OIT approximation used to describe the kinetics of particles obeying an EP, Eq. (25) can also be written in the form given by Eq. (55) if a potential $U(t,v,p)$, depending on $p(v,t)$ defined by

$$\frac{\partial U}{\partial v}=\frac{1}{D}\left[\frac{\partial D}{\partial v}-2J\right](1-p) , \tag{56}$$

is introduced.

We are now looking for the expression of U when $t\rightarrow\infty$. Let us define

$$U(v)=U(\infty,v,p(\infty,v)) . \tag{57}$$

Substituting $p(\infty,v)$, given by Eq. (45) (the distribution function of Brownian classical particles obeying the EP), into Eq. (56) we obtain

$$\frac{\partial U(v)}{\partial v}=\frac{m}{kT}\frac{v\exp[(\frac{1}{2}mv^2-\mu)/kT]}{1-\exp[(\frac{1}{2}mv^2-\mu)/kT]} . \tag{58}$$

The potential defined by Eq. (58) has the following form:

$$U(v)=U_0(v)+U_{\rm EP}(v) , \tag{59}$$

where $U_0(v)$ is the parabolic potential of a Brownian particle which does not obey the EP and is given by the expression

$$U_0(v)=\frac{m}{2kT}v^2+\text{const} . \tag{60}$$

The second term in the right-hand side of Eq. (59) is given by

$$U_{\rm EP}(v)=-\frac{m}{kT}\int\frac{vdv}{1+\exp[(\frac{1}{2}mv^2-\mu)/kT]}+\text{const} \tag{61}$$

and is due to the EP.

After integration, we have

$$U_{\rm EP}(v)=\ln\{1+\exp[-(\tfrac{1}{2}mv^2-\mu)/kT]\}+\text{const} . \tag{62}$$

The potential U_{EP} is repulsive, finite for $-\infty < v < \infty$, and goes to the value of the constant of Eq. (62) for $v \to \pm\infty$. The sum of U_0 and U_{EP} gives an attractive potential $U(v)$, which is no longer parabolic, does not go to zero for $v \approx 0$, and shows a repulsion in order to prevent the collapse at the lowest level. We can also say that the EP potential explicitly expressed in the case of Brownian particles by Eq. (62) is strictly related to the exact Fermi-Dirac distribution of Eq. (45). In the other situations, unlike the Brownian case, the EP potential can be expressed through an integral form.

Dorso, Duarte, and Randrup [30] developed a model based on the microscopic dynamics of interacting classical particles. They introduced a momentum-dependent potential for the purpose of simulating the Pauli exclusion principle; this potential also includes a certain type of uncertainty relation.

Wilets and co-workers [27–29], in different studies on heavy-ion collisions, postulated a momentum-dependent potential with the parameters of the two-body nuclear potential and found that it was possible to fit some nuclear properties. However, they never demonstrated that the phase-space distribution of the nucleon is actually well approximated. Dorso, Duarte, and Randrup [30] determined an appropriate Pauli potential and demonstrated that one can obtain a good reproduction of important features of the Fermi gas if one choses appropriate parameters. Their Pauli potential prevents the particles from assembling at zero momentum as they would in the absence of the momentum-dependent repulsion and produces a phase-space occupation that is close to the correct quantum-mechanical one. A comparison of the two different potentials introduced by Dorso, Duarte, and Randrup [30] (of Gaussian form) and by us [Eq. (62)] shows that the two potentials have a close behavior at low values of v, which can become very close with particular choices of the parameters of both potentials.

VIII. CONCLUSIONS

A generalized Fokker-Planck equation has been proposed which describes, in a one-dimensional heterogeneous and anisotropic space, the kinetics of classical particles, obeying an exclusion principle, such as the Pauli one. To derive this equation, which can be seen as the limit to the continuum of a master equation of a Markovian process, we have considered the particle transitions occurring at the same time (the CT description). In the approximation of only individual transitions and when the space is homogeneous, this equation equals the Burgers equation. One can conclude after the investigation of the proposed equation that the main results obtained in this work are the following.

The OIT approximation cannot describe the random walk (diffusion in a homogeneous and isotropic space) of particles obeying an EP. The simple hypothesis of the individual transitions describes a kinetics which is not affected by the inclusion of the EP. We show that in the CT description, introduced in this paper, it is possible to consider correctly the EP in the random-walk kinetics. We obtain that this kinetics is equivalent to nonlinear diffusion with the diffusion coefficient being a third-order polynomial.

If we identify the space with the velocity space and if we consider Brownian particles, we obtain that the steady state of the proposed equation is equal, in the OIT approximation, to the FD distribution.

This result is very peculiar because the particles here considered are classical while the FD distribution defines a quantum statistics. However, it has been shown by Wilets and co-workers and later by Dorso, Duarte, and Randrup that one can introduce an exclusion principle into the classical equations by means of a potential, and with specific values of the parameters, one can derive for the Fermi gas a distribution very close to the FD one.

Our approach is equivalent to the introduction of a potential which simulates the EP, and we have verified that our potential gives the exact FD distribution for Brownian particles.

In the CT description, which rigorously takes the EP into consideration, we obtain a alternative statistical distribution differing slightly from that of the FD. These results support our approach, which will be extended and applied in a subsequent paper to many different physical systems.

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